<u>Recall:</u> we showed that any vector space over any field has a basis.

Invariance of Dimension

Definition: (dimension)

Given a vector space V over a field $\mathbb{F}$ and a basis $\mathcal{B}$ for V over $\mathbb{F}$, define the dimension of V over $\mathbb{F}$ to be the cardinality of $\mathcal{B}$.

Example 1 :

The dimension of $\mathbb{F}^n$ over $\mathbb{F}$ is n since placing $1_{\mathbb{F}}$ in the i^{th} position and $0_{\mathbb{F}}$ in all other positions gives a linearly independent and spanning set.

Example 2:

The dimension of $\mathbb{C}$ over $\mathbb{C}$ is 1, but the dimension of $\mathbb{C}$ over $\mathbb{R}$ is two since a basis is $\{1, i\}$.

Example 3:

The dimension of the space of all polynomials with real coefficients (over $\mathbb{R}$) is countably infinite since a basis is given by

$$\{x^n\}_{n=0}^{\infty}.$$

Example 4:

The dimension of $\mathbb{R}$ over $\mathbb{R}$ is one, but the dimension of $\mathbb{R}$ over $\mathbb{Q}$ is infinite – in fact it is uncountably infinite!

For example,

$\{\pi^n\}_{n=1}^{\infty}$ is linearly independent over $\mathbb{Q}$

since if $\exists$ $n \in \mathbb{N}$ and

$\alpha_1, \alpha_2, -, \alpha_n \in \mathbb{Q}$

not all zero with

$\sum_{k=1}^{n} \alpha_k \pi^k = 0$, then

π is a zero of

$p(x) = \sum_{k=1}^{n} \alpha_k x^k$.

$\forall\ k,\ \exists\ a_k, b_k \in \mathbb{Z},$

$b_k \neq 0$, with

$\alpha_k = \dfrac{a_k}{b_k}$. Then

multiplying $p(x)$ by

$b_1 \cdot b_2 \cdot \cdots \cdot b_k$, we clear

denominators and obtain

a polynomial $q(x)$.

Now since $p(\pi)=0$,

$$q(\pi) = (b_1 \cdot b_2 \cdot \dots \cdot b_n)\, p(\pi)$$

$= 0$, and so

π would be a root of a polynomial with integer coefficients. But π is transcendental, so it is not the root of any such polynomial, contradiction.

Note that a basis for $\mathbb{R}$ over $\mathbb{Q}$ cannot be countable since if $\{f_n\}_{n=1}^{\infty}$ were such a basis, let

$Q_k =$

$$\left\{x \in \mathbb{R} \mid \exists\, \alpha_1, \ldots, \alpha_k \in \mathbb{Q},\ x = \sum_{n=1}^{k} \alpha_n f_n\right\}$$

Then each Q_k is countable and we obtain

$$\mathbb{R} = \bigcup_{k=1}^{\infty} Q_k$$

Since we can write every element in $\mathbb{R}$ as a finite linear combination of the f_n's.

But a countable union of countable sets is countable and $\mathbb{R}$ is uncountable, contradiction.

Q: Why is this well-defined?

i.e. how do we know, given any two bases of V over $\mathbb{F}$, say B_1 and B_2, that

$$|B_1| = |B_2|?$$

Ingredients for the Proof:

1) Axiom of Choice (equivalent to the Hausdorff Maximality Principle): Given any collection of sets $(X_i)_{i \in I}$, $\exists$ a function

$$f : \{X_i\}_{i \in I} \to \bigsqcup_{i \in I} X_i$$

("$\sqcup$" = disjoint union)

such that

$f(X_i) = y_i \in X_i$

$\forall\, i \in I$, i.e.,

f picks out a single element from each set.

2) Schröder-Bernstein:

Given two sets X and Y,

if $\exists$ injections

$f: X \to Y$ and

$g: Y \to X$, then

$\exists$ a bijection $h: X \to Y$,

so $|X| = |Y|$.

(HW #3)

Lemma: Let S be any set and suppose that $|S|$ is infinite. Then if we decompose

$$S = \bigsqcup_{i \in I} S_i$$

with $S_i \subseteq S \; \forall \, i \in I$ and each S_i is finite, then $|S| = |I|$.

Proof: extra credit!

I only want to see complete solutions.

Proof that any two bases have the same cardinality

(Jacobson, "Lectures in Abstract Algebra $\underline{\mathrm{II}}$")

Let $B_1 = \{e_i\}_{i \in I}$ and

$B_2 = \{f_j\}_{j \in J}$ be

two basis for V over $\mathbb{F}$.

For each e_i,

$\exists\ \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}$

and $f_{j_1}, f_{j_2}, \ldots, f_{j_n} \in B_2$

with $e_i = \sum_{k=1}^{n} \alpha_k f_{j_k}$

(since $B_2 = \{f_j\}_{j \in J}$ is a basis)

Claim: If $j \in \bar{J}$, then $\exists\ i \in I$ such that f_j occurs as a nonzero term in the expansion of e_i in the basis B_2.

Proof by contradiction:

Suppose f_j does not appear in any expansion of the e_i's.

Since $\{e_i\}_{i \in I}$ is a basis, we can find scalars $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{F}$ and $e_{i_1}, e_{i_2}, \dots, e_{i_m} \in \mathcal{B}$, such that

$$f_j = \sum_{t=1}^{m} \beta_t e_{i_t}.$$

But for each i_t we can expand

e_{i_t} as a linear

combination of elements

in B_2, none of which are equal to f_j by assumption.

But then f_j is a linear combination of elements in B_2, none of which are equal to f_j. This contradicts linear independence of B_2, so $\forall\ j \in J$, f_j occurs in the expansion of some e_i.

Define $\varphi : B_2 \to B_1$

by $\varphi(f_j) = e_{i(j)}$

where f_j occurs in the expansion of $e_{i(j)}$ (axiom of choice).

Since each f_j occurs in the expansion of some e_i, this definition is not vacuous.

The preimage $\varphi^{-1}(e_{i(j)})$ consists of a subset of the vectors in the expansion of $e_{i(j)}$. These are disjoint subsets of B_2 which are each finite (any expansion of $e_{i(j)}$ is finite).

Moreover, since φ is defined on all of B_2, the union of all preimages is equal to B_1. By selecting a single element in each preimage (axiom of choice), we get an injection from a subset of B_1 to the disjoint union. But by the lemma, the cardinality of the union is equal to the cardinality of B_2, so we have an injection

$$\varphi: B_2 \to B_1.$$

Reversing the roles of B_1 and B_2, we obtain an injection

$$\psi : B_1 \to B_2 .$$

Then by Schröder-Bernstein, $|B_1| = |B_2|$. Hence, dimension is well-defined!

$\square$